COVARIANT SINGLE-TIME BOUND-STATE EQUATION [1]

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Abstract

We derive a system of covariant single-time equations for a two-body bound state in a model of scalar fields ϕ_1 and ϕ_2 interacting via exchange of another scalar field χ. The derivation of the system of equations follows from the Haag expansion. The equations are linear integral equations that are explicitly symmetric in the masses, m_1 and m_2, of the scalar fields, ϕ_1 and ϕ_2. We present an approximate analytic formula for the mass eigenvalue of the ground state and give numerical results for the amplitudes for a choice of constituent and exchanged particle masses.

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1. INTRODUCTION

The problem of relativistic bound states has a long history. Nonetheless, the treatment of this problem is still not completely satisfactory. The purpose of this paper is to continue the development of an alternative to the most popular formulation, the Bethe-Salpeter method[1]. The Bethe-Salpeter method uses amplitudes in which both constituents are off-shell. Because of this, the amplitudes depend on an unphysical relative-time coordinate and obey equations that are difficult to solve and have spurious unphysical solutions, including some of negative norm. Several authors have proposed covariant, single-time equations with only one constituent off-shell. The equations most similar to ours are the "spectator equations" of F. Gross[2]. Our equations differ from the spectator equations in the way we ensure symmetry between the off-shell and on-shell particles, in the inclusion of renormalization graphs and counter terms and in the boundary conditions of the Green's functions: our equations use Green's functions with retarded boundary conditions, rather than Feynman boundary conditions. Our derivation of the equations differs entirely from Gross' derivation of the spectator equation: we use the Haag expansion[3] and the operator field equations[4, 5, 6, 7, 8], rather than summing classes of Feynman graphs. To be concrete, we consider a two-body bound state in a model of scalar fields ϕ_1 and ϕ_2 interacting via exchange of another scalar χ. A. Raychaudhuri used this method to study bound states in the equal-mass case of this model ; however his equations are not symmetric in the on-shell and off-shell masses[9]. This asymmetry was not evident in the equal-mass case. He also studied the nonrelativistic reduction of the equations for unequal-mass bound states of spin-1/2 particles[10]. Related work was done by M. Bander, et al[11].

In this paper, we extend Raychaudhuri's analysis to unequal-mass constituents and treat the on-shell and off-shell particles in a completely symmetric way. We present numerical results for the ground state eigenvalue and amplitudes for a range of m_1/m_2 and μ/m_2, where μ is the mass of the χ field.

We hope this method will provide an alternative to the Bethe-Salpeter method with the following advantages: (1) because only one particle at a time is off-shell, the amplitudes depend on only one invariant, (2) all normalizable solutions are physical and have positive norm, (3) the limit for one mass very large is the relativistic equation for the other particle bound in an external field and (4) the nonrelativistic limit has the correct reduced mass. We have two longer-range goals: (1) to extend

the Haag expansion to account for cases in which the interaction is strong enough to make a two-particle description inadequate and (2) to modify the Haag expansion to treat confined degrees of freedom, for which the usual asymptotic fields don't exist. This latter goal will require a significant generalization of the method.

2. DERIVATION OF THE EQUATIONS

In this section we obtain coupled integral equations for two bound-state amplitudes, one, f_1, with particle one off-shell and particle two on-shell, the other, f_2, with these roles interchanged. Our Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{2} \frac{1}{2}(\partial_\mu \phi_i \partial^\mu \phi_i) - m_i^2 \phi_i^2) + \frac{1}{2}(\partial_\mu \chi \partial^\mu \chi - \mu^2 \chi^2) + \frac{g}{4}[\phi_1^2 + \phi_2^2, \chi]_+, \qquad (1)$$

where the last term is an anticommutator. We work in momentum space using $\phi(x) = (2\pi)^{-3/2} \int d^4p \tilde{\phi}(p) exp(-ip \cdot x)$ and the analogous formula with in fields. We promptly drop the tilde on ϕ, abbreviate d^4p by dp and, for the in field, abbreviate $: \phi^{in}(p)\delta(p^2 - m_i^2) :$ by $: \phi^{in}(p) :$. The equations of motion in momentum space are

$$(m_i^2 - p^2)\phi_i(p) = \frac{g}{2(2\pi)^{3/2}} \int dp_1 dp_2 \delta(p - p_1 - p_2)[\phi_i(p_1), \chi(p_2)]_+ \qquad (2)$$

$$+(A_i p^2 - B_i m_i^2)\phi_i(p) \qquad (3)$$

$$(\mu^2 - p^2)\chi(p) = \frac{g}{2(2\pi)^{3/2}} \sum_{i=1}^{2} \int dp_1 dp_2 \delta(p - p_1 - p_2)\phi_i(p_1)\phi_i(p_2) \qquad (4)$$

$$+(Dp^2 - E\mu^2)\chi(p), \qquad (5)$$

where we have introduced counter terms for the mass and field strength renormalizations of the fields.

In the N-quantum approximation, we expand the Lagrangian fields in terms of the complete, irreducible set of in-fields (or out-fields), including those for stable bound states (this is the Haag expansion), and truncate the expansion to find an approximate set of equations among a finite number of amplitudes. Here we keep all terms that contribute to equations for the two-body bound-state amplitudes f_1 and f_2 mentioned above in one-loop approximation. All the terms have explicit order g^2 at the perturbative vertices. The relevant terms in the Haag expansion are

$$\begin{aligned}\phi_1(p) &= :\phi_1^{in}(p): + \int dqdb\delta(p+q-b)f_1(q,b) :\phi_2^{in}(-q)B^{in}(b): \\ &+ \int dqdldb\delta(b-q-l-b)f_{\chi 2B}(q,b,l) :\chi^{in}(q)\phi_2^{in}(l)B^{in}(b):,\end{aligned}$$

$$\begin{aligned}\chi(p) &= :\chi^{in}(p): + \int dl_1 dl_2 \delta(p-l_1-l_2)\gamma_{ii}(l_1,l_2) :\phi_i^{in}(l_1)\phi_i^{in}(l_2) \\ &+ \int dl_1 dl_2 db \delta(p-l_1-l_2-b)\gamma_{12B}(l_1,l_2,b) :\phi_1^{in}(l_1)\phi_2^{in}(l_2)B^{in}(b):,\end{aligned}$$

and the analogous terms for ϕ_2, with 1 and 2 interchanged. Our general notation is that the subscript on an amplitude lists its associated product of in fields. What we call f_1 should be f_{2B} according to this general notation; however, for convenience, we call it f_1. Because each in field has a mass shell δ-function, all the momentum integrals in the Haag expansion are on two-sheeted mass hyperboloids. In f_1 we keep b on the positive-energy mass shell and reverse the sign of the momentum q in f_1 so that q on the positive-energy hyperboloid gives the dominant amplitude in the nonrelativistic limit. We call $f_1(q,b)$ with $q > 0$, i.e. with q on the positive mass hyperboloid, $f_1^{(+)}$ and with $q < 0$, $f_1^{(-)}$. (See Fig. 1) Both our equations and their graphical representation include both of these pieces of the amplitudes; to save space we don't exhibit both pieces in the graphs.

As usual, $::$ denotes normal ordering. In the one-loop approximation, contractions always involve the vacuum matrix element of the anticommutator,

$$\begin{aligned}\langle[\phi_i^{in}(p_1),\phi_j^{in}(p_2)]_+\rangle_0 &= \delta_{ij}\delta(p_1+p_2)\delta_{m_i}(p_1), \\ \langle[\chi^{in}(p_1),\chi^{in}(p_2)]_+\rangle_0 &= \delta(p_1+p_2)\delta_\mu(p_1),\end{aligned}$$

where $\delta_m(p) = \delta(p^2 - m^2)$ for short. Choosing to expand the Lagrangian fields in terms of the in-fields requires using retarded boundary conditions for the propagators.

To obtain the equation for f_1, we insert the Haag expansions for ϕ_1 and χ in the equation of motion for ϕ_1, renormal order and equate the coefficients of the term with $:\phi_2^{in}B^{in}:$. The resulting equation involves the amplitudes f_1, $f_{\chi 2B}$, γ_{22}, and γ_{12B}. We calculate the last three amplitudes in terms of f_1 using the equations

of motion and the Born approximation for emission of both on-shell and off-shell χ quanta. We will give details of this in a later, more detailed paper.

$$f_{\chi 2B}(q,l,b) \;=\; \frac{g}{(2\pi)^{3/2}} \frac{f_1(-l,b)}{m_1^2-(q+l+b)^2}, \tag{6}$$

$$\gamma_{11}(p_1,p_2) \;=\; \gamma_{22}(p_1,p_2) = \frac{g}{2(2\pi)^{3/2}} \frac{1}{\mu^2-(p_1+p_2)^2}, \tag{7}$$

$$\gamma_{12B}(l_1,l_2,b) \;=\; \frac{g}{(2\pi)^{3/2}} \frac{f_1(-l_2,b)+f_2(-l_1,b)}{\mu^2-(l_1+l_2+b)^2}. \tag{8}$$

$$\tag{9}$$

The integral equation for f_1 is

$$\begin{aligned}
&[m_1^2-(b-p)^2]f_1(p,b) = \\
&\frac{g^2}{16\pi^3}\int dq\left[\frac{\delta_{m_1}(q)}{\mu^2-(b-p-q)^2}+\frac{\delta_{\mu}(q)}{m_1^2-(b-p-q)^2}\right]f_1(q,b) \\
&+\frac{g^2}{16\pi^3}\int dq\left[\frac{\delta_{m_2}(q)}{\mu^2-(p-q)^2}f_1(q,b)+\frac{\delta_{m_1}(q)}{\mu^2-(b-p-q)^2}f_2(q,b)\right] \\
&+[A_1(b-p)^2-B_1m_1^2]f_1(p,b),
\end{aligned} \tag{10}$$

where the first two terms on the right hand side are self-energy graphs that are completely canceled by the renormalization counter terms. Any method of regularization of the self-energy graphs will suffice. The third and fourth terms on the right give binding by exchange of the χ field. The bound-state momentum b is always on its mass shell $b^2 = M^2$. This bound-state equation is shown in Fig. 2. We get another coupled equation for f_2 by interchanging 1 and 2. The resulting pair of equations is clearly symmetric under 1, 2 interchange. We suppress the $i\epsilon$'s associated with the retarded boundary conditions.

3. Approximate Mass Eigenvalue Formula

We considered parametrizing the mass eigenvalue formula using the $\arccos\eta$, where $\eta = M/(m_1+m_2)$, because this expression appears in the hyperboloidal harmonic analysis that Raychaudhuri[9] used. We found interesting empirical regularities using this parametrization. The result is

$$M = (m_1+m_2)\cos\frac{\lambda-a}{b}, \tag{11}$$

where $\lambda = g^2/(32\pi m_1 m_2)$, $a = 0.9\sqrt{\mu/m_{red}}$ and $b = 0.8 - 1.1\ \ln(m_</m_>)$. The reduced mass is the usual expression; $m_>$ is the larger of m_1 and m_2. The range of validity of this empirical formula is $0 \leq \mu \leq m_<$, $0.01 \leq m_</m_> \leq 1$, $0.5 \leq \eta \leq 1$ for $m_</m_> = 1$ and $0.9 \leq \eta \leq 1$ for $m_</m_> = 0.1$.

4. Numerical Results

Equation (9) and the one with m_1 and m_2 interchanged are eigenvalue equations for the coupling constant g; that means for given values of the masses M, m_1, m_2 and μ we can find a coupling constant g and wave functions $f_{1,2}^{(\pm)}$ that satisfy the equations. We solve these homogeneous linear integral equations by approximating the integral on the right hand side with a finite sum. We choose Gauss integration with appropriate points and weights. The resulting matrix equation is solved by standard means.

For the equation in momentum space it is sufficient to take 18 mesh points to obtain g^2 to an accuracy of 4%. The main difficulties we encounter in Eq. (9) are the logarithmic singularities. We smooth these singularities by keeping a finite ϵ at the logarithmic singularity. We checked that the result does not change by varying the mesh points and ϵ.

In Fig. 3 the value of $\lambda = g^2/(32\pi m_1 m_2)$ is plotted as a function of $\eta = M/(m_1 + m_2)$ for $m_</m_> = 0.1$ and $\mu = 0$. A calculation using the Bethe -Salpeter equation [12] consistently gives smaller binding. In the scalar model we cannot decide which solution is correct because there is no experimental data. Figure 4 shows the wave functions $f_1^{(+)}$, $f_1^{(-)}$, $f_2^{(+)}$ and $f_2^{(-)}$, respectively, for the mass ratios given above and for $\eta = 0.95$. As expected $f_1^{(+)}$ is the dominant contribution.

We will present more extensive numerical results in a later paper.

5. Summary and outlook for future work

The Haag expansion leads directly to coupled linear integral equations for four amplitudes related to a scalar bound state. Two amplitudes are those that reduce to the nonrelativistic wavefunction, one, f_1, with the particle of mass m_1 off-shell and the other, f_2, with the particle of mass m_2 off-shell. The other two amplitudes have the on-shell particle crossed, so that its momentum lies in the same light cone as the bound-state momentum. These four amplitudes obey a set of four coupled linear integral equations. We solved these numerically using momentum-space variables.

We plan to apply this method to bound states of two spin-1/2 particles, such as the hydrogen atom and positronium, where our calculations can be compared with experimental results. We hope this method can replace the Bethe-Salpeter method in theories without confinement.

In order to use this method in confining theories, such as QCD, the asymptotic fields that are a prominent part of the Haag expansion must be replaced with fields that correspond to confined degrees of freedom. The treatment of confined degrees of freedom in theories such as QCD remains a goal for the future.

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Captions

Fig. 1: The two pieces of f_1. The short line through a leg indicates the leg is off-shell.

Fig. 2: Graphs for the bound state equation for f_1 if the left-hand leg is ϕ_1. Note that the first two terms are self-energy graphs that are canceled by the counter terms, the third term is a t-channel graph that couples f_1 to itself and the last term is a u-channel graph that couples f_1 to f_2.

Fig. 3: Plot of $\lambda = g^2/(32\pi m_1 m_2)$ as a function of $\eta = M/(m_1+m_2)$ for $\mu = 0,\ m_</m_> = 0.1$.

Fig. 4: Wave functions $f_1^{(+)}$, $f_1^{(-)}$, $f_2^{(+)}$ and $f_2^{(-)}$ in momentum space in arbitrary units as a function of Λ for $\eta = 0.95$, $\mu = 0$ and $m_</m_> = 0.1$. Here $m_> \cosh\Lambda = \sqrt{\mathbf{p}^2 + m_>^2}$.

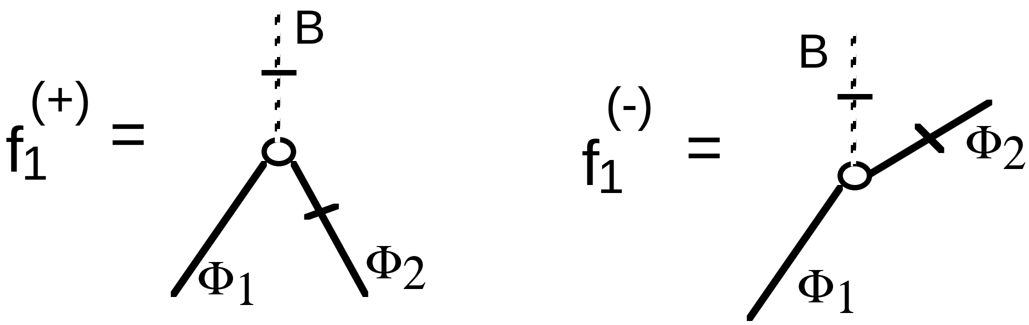


Fig. 1

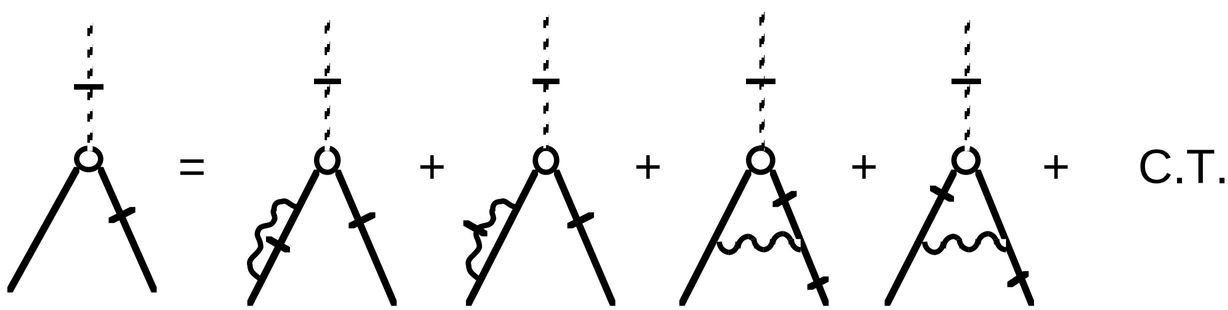


Fig. 2

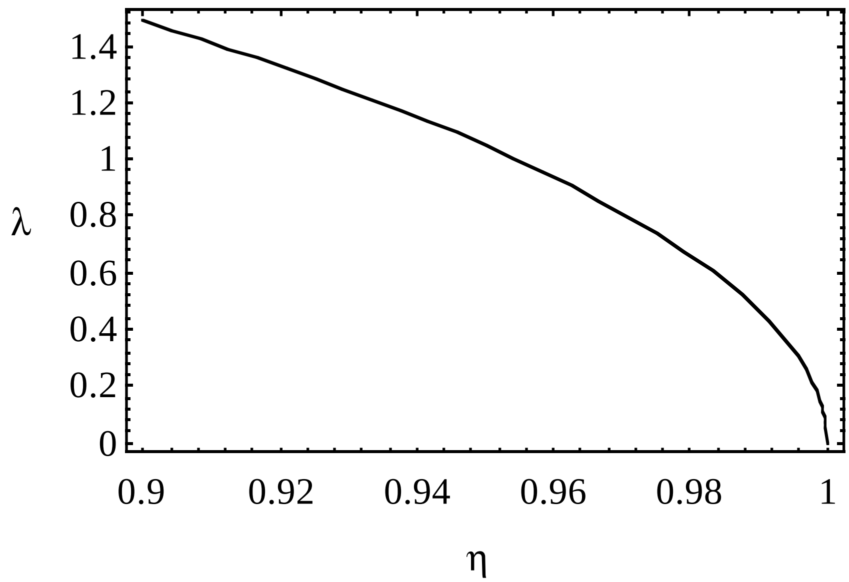
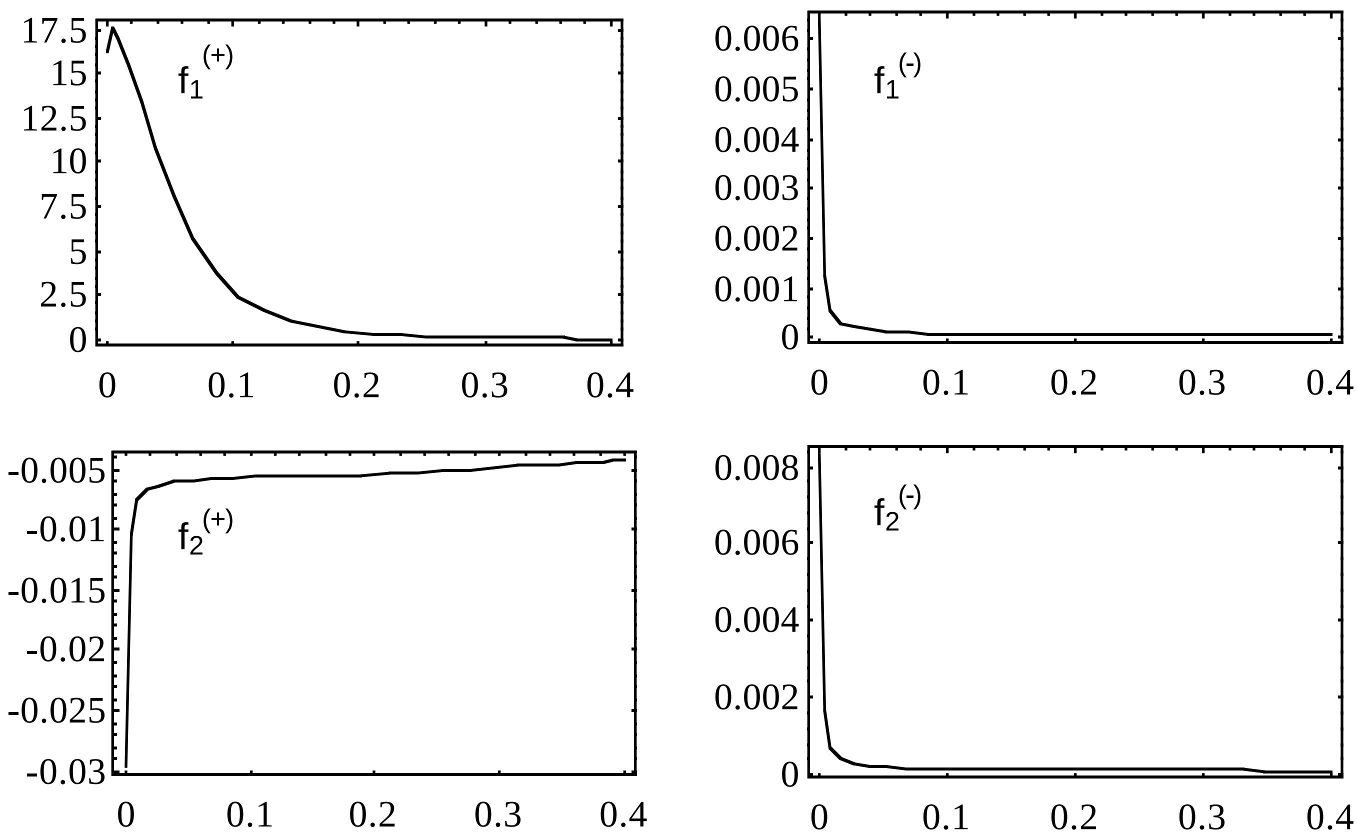


Fig. 3



17.5
15
12.5
10
7.5
5
2.5
0
0
0.1
0.2
0.3
0.4
$f_1^{(+)}$
0.006
0.005
0.004
0.003
0.002
0.001
0
$f_1^{(-)}$
-0.005
-0.01
-0.015
-0.02
-0.025
-0.03
$f_2^{(+)}$
0.008
0.006
0.004
0.002
$f_2^{(-)}$

Fig. 4